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Orthogonal X waves

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Abstract
Nondiffracting pulses are spatially and temporally localized wave fields that undergo no diffractive spreading under propagation through homogeneous media. We introduce an orthogonality condition for nondiffracting pulses and present an orthogonal set of X waves which possess temporal spectra of the form (polynomial in ω) $\times$ $e^{-\alpha\omega}$. The newly introduced Bessel-X pulses and X-wave transforms are discussed in the framework of the orthogonal X-wave bases.

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1. Introduction

Orthogonal representations are essential in mathematical physics and they often provide deep insight into physical phenomena. Nondiffracting beams [1] and pulses [2] are transverse-localized wave fields which propagate in free space (or homogeneous media) without changing their spatial profile. Moreover, all ideal nondiffracting waves feature superluminal phase—and pulse—propagation without violating relativistic causality [3].

Nondiffracting waves are also called X waves, since their intensity pattern resembles the letter ‘X’ in the meridional plane. In fact, the superluminal pulse only results from the superposition of the conical wave fronts, limiting the signal velocity to that of light in vacuum. Optical generation of nondiffracting waves is well-established [1, 4, 5], and new wave modes are actively being reported [6, 7].

The first optical realization of the so-called Bessel-X pulses by Saari and Reivelt [8] led to the idea of using the X waves in optical communication [9], which also could take advantage of different orthogonal X-wave modes as separate communication channels. Additionally, the new X-wave transform [10], which provides a general representation of all the waves in terms of nondiffracting waves of different cone angles, crucially depends on the orthogonality of the individual wave modes.

In this paper, we present an orthogonality condition for nondiffracting waves and apply it to derive an orthogonal basis for the X waves, i.e., nondiffracting pulses that are linear combinations of waves having Fourier spectra of the form $\omega^m e^{-\alpha\omega}$ [11–13]. We also employ orthogonal waves to represent Bessel-X pulses and discuss their physical properties.

2. Spectral X-wave representation

Nondiffracting waves are defined through the condition of uniform propagation, $\Phi(r, \varphi, z; t) = \Phi(r, \varphi, z - vt)$ where (r, φ, z) denote cylindrical spatial coordinates and v is the velocity of propagation for the wave field. This condition leads to the Fourier representation of a nondiffracting wave, given by [13]

$$\tilde{\Phi}(\boldsymbol{k}, \omega) = F(\omega, \beta)\delta(k_\perp - \omega \sin\zeta/c)\delta(k_z - \omega \cos\zeta/c) \tag{1}$$

where $F(\omega, \beta)$ is an arbitrary function and $(k_\perp, \beta, k_z)$ denote cylindrical coordinates in the wave vector space. The cone angle is defined via $\cos\zeta = c/v$. The axial and radial wave vector components, k_z and $k_\perp$, are uniquely defined for each frequency ω, and the wave vectors constitute a cone which opens from the k_z-axis in the angle extended through ζ (hence the name cone angle). Nondiffracting waves in the real space are obtained via inverse Fourier transformation:

$$\Phi(\boldsymbol{r}, t) = \frac{1}{(2\pi)^2}\int_0^\infty d\omega \int_{-\infty}^\infty dk_z \int_0^\infty k_\perp\, dk_\perp \times \int_0^{2\pi} d\beta\, \tilde{\Phi}(\boldsymbol{k}, \omega)\, e^{i(xk_\perp\cos\beta + yk_\perp\sin\beta + zk_z - \omega t)}. \tag{2}$$

We deliberately choose to limit our discussion to complex analytic signals, thus constraining the temporal spectrum to positive frequencies only. Use of the Fourier series

$$F(\omega, \beta) = \sum_{n=-\infty}^{\infty} (-i)^n (2\pi/k_\perp) f_n(\omega)\, e^{in\beta} \tag{3}$$

leads to the general integral representation for nondiffracting waves of azimuthal order n:

$$\Phi_n(\boldsymbol{r}, t) = e^{in\varphi}\int_0^\infty f_n(\omega) J_n(r(\omega/c)\sin\zeta)\, e^{i(z\cos\zeta/c - t)\omega}\, d\omega. \tag{4}$$

Above, the complex exponential function defines the azimuthal order of the nondiffracting wave, and the normalization factor $(-i)^n 2\pi/k_\perp$ is added in equation (3) to cancel all the constant factors in the above expression. This is the general representation for nondiffracting waves with a fixed azimuthal order n and every nondiffracting wave can be obtained from waves of this form by suitably weighing and summing over n. We call the function $f_n(\omega)$ the Fourier spectrum of the nth wave mode.

3. Orthogonality of nondiffracting waves

We now turn to consider the orthogonality condition for nondiffracting waves. Taking two wave fields at a fixed time t, the ordinary L_2 scalar product is defined as

$$\langle \Phi_F \mid \Phi_G \rangle = \int_{-\infty}^\infty \Phi_F^*(x, y, z; t)\Phi_G(x, y, z; t)\, dx\, dy\, dz. \tag{5}$$

Here Φ_F and Φ_G represent two nondiffracting waves with the spectral functions $F(\omega, \beta)$ and $G(\omega, \beta)$ and with the cone angles ζ_F and ζ_G, respectively. By inserting their Fourier representations into equation (5), their scalar product is given by (see the appendix for details and discussion) the expression

$$\langle \Phi_F | \Phi_G \rangle = \frac{c}{2\pi \cos \zeta_G} \left| 1 - \frac{\tan \zeta_G}{\tan \zeta_F} \right|^{-1} \int_0^{2\pi} \mathrm{d}\beta [|\omega| F^*(\omega, \beta) G(\omega, \beta)]_{\omega=0} \tag{6}$$

for waves with different cone angles. The scalar product between waves of different velocities always has a finite value, provided that the spectral function does not diverge at zero frequency. Note that this does not lead to a norm since it cannot be evaluated for a wave with itself. If two waves share the same cone angle (velocity), they are orthogonal provided that

$$\int_0^\infty \mathrm{d}\omega \int_0^{2\pi} \mathrm{d}\beta\, \omega\, F^*(\omega, \beta) G(\omega, \beta) = 0. \tag{7}$$

If, however, this integral is not zero, the scalar product diverges, reflecting the fact that nondiffracting waves have infinite L_2 norm, or, equivalently, they carry an infinite amount of energy.

Using a series representation (equation (3)) for the spectral functions F and G, the orthogonality condition may be cast into the form

$$\sum_{n=-\infty}^{\infty} \int_0^\infty f_n^*(\omega) g_n(\omega) \omega^{-1}\, \mathrm{d}\omega = 0. \tag{8}$$

Consequently, waves of different azimuthal orders n are always mutually orthogonal. The space of nondiffracting waves consists of orthogonal subspaces which are characterized by their azimuthal order, n.

4. Orthogonal X-wave basis

We define elementary X waves by choosing the specific spectral form given by (see [13] for an extensive discussion)

$$f_n(\omega) = \omega^m\, \mathrm{e}^{-\alpha\omega}. \tag{9}$$

The spectrum contains two parameters: (i) the spectral order m which assumes positive integer values, or zero, and (ii) the spectral attenuation factor α. Nondiffracting waves corresponding to the spectrum of the form in equation (9) have closed-form algebraic expressions

$$\Phi_{n,m}(\boldsymbol{r}, t) = (-1)^{*n}\, \mathrm{e}^{\mathrm{i}n\varphi} \frac{\Gamma(m + |n| + 1)}{\left(\sqrt{M}\right)^{m+1}} \left(\sqrt{\frac{1 - Q}{1 + Q}} \right)^{|n|} \times \sum_{k=0}^{m} (-1)^k \frac{(m + k)!/(m - k)!}{(|n| + k)!} \frac{(1 - Q)^k}{2^k k!} \tag{10}$$

where $M = \tau^2 + \beta^2$ and $Q = \tau/\sqrt{M}$ with $\tau = \alpha - \mathrm{i}[(\cos\zeta)z/c - t]$ and $b = r(\sin\zeta)/c$, while $(-1)^{*n} = 1$ for positive n and $(-1)^n$ for negative n. The whole set of X waves is defined to consist of linear combinations

$$\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \Phi_{n,m} \tag{11}$$

where the $c_{n,m}$ serve as complex weight functions for each elementary X wave. The $\Phi_{n,m}$ constitute a set of basis functions for all X waves but they do not feature any particular normalization or orthogonality properties.

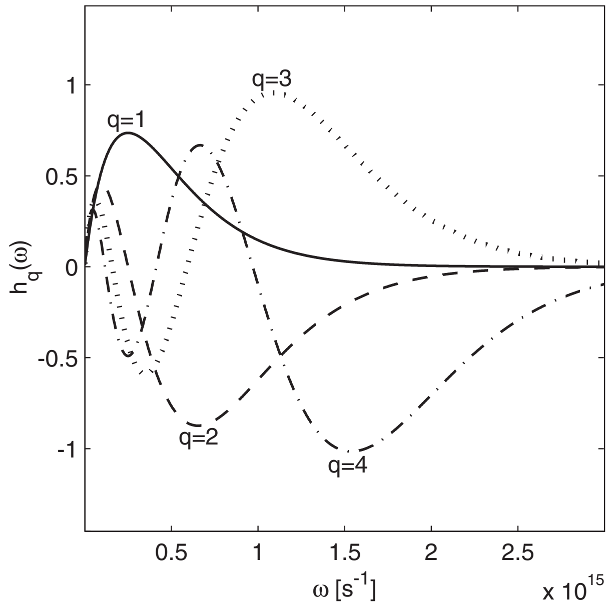

Figure 1. Orthogonal spectra of orders 1 (solid line), 2 (dashed line), 3 (dotted line) and 4 (dashed–dotted line). Spectral functions have been evaluated for $\alpha = 4$ fs.

We now turn to look for an orthogonal basis for nondiffracting waves. We require the basis functions to be of a uniquely defined Bessel order, whence the spectra of two basis functions (of the same Bessel order) have to satisfy

$$\int_0^\infty f_n^*(\omega) g_n(\omega) \omega^{-1} \, \mathrm{d}\omega = 0. \tag{12}$$

Prior to proceeding to the X waves, here we want to emphasize that equation (12) does not represent a regular scalar product since it has a divergent $1/\omega$ factor. Therefore, we limit our consideration to such spectra for which the above expression has a finite value.

4.1. Orthogonal X waves

Since all nondiffracting waves can be uniquely separated into a sum of waves with different azimuthal orders n, the same division also holds for the X waves. We therefore consider waves given by equation (4) whose spectra have the X-wave form

$$f_n(\omega) = p(\omega) \mathrm{e}^{-\alpha\omega} \tag{13}$$

where $p(\omega)$ is a polynomial of arbitrary degree. Two such waves are orthogonal provided that

$$\int_0^\infty p_1(\omega) p_2(\omega) \omega^{-1} \, \mathrm{e}^{-2\alpha\omega} \, \mathrm{d}\omega = 0. \tag{14}$$

This condition is satisfied for

$$f_n(\omega) = h_q(\omega) = \frac{2\alpha\omega}{\sqrt{q+1}} L_q^{(1)}(2\alpha\omega) \, \mathrm{e}^{-\alpha\omega} = \sum_{l=0}^{q} \frac{(-1)^l}{\sqrt{q+1}} \binom{q+1}{q-l} \frac{(2\alpha)^{l+1}}{l!} \omega^{l+1} \, \mathrm{e}^{-\alpha\omega} \tag{15}$$

where $L_q^{(1)}$ are generalized Laguerre polynomials (see figure 1). The orthogonality of the spectra follows directly from the orthogonality properties of Laguerre polynomials:

$$\int_0^\infty h_q(\omega)h_p(\omega)\omega^{-1}\,\mathrm{d}\omega = \frac{1}{\sqrt{(q+1)(p+1)}}\int_0^\infty L_q^{(1)}(s)L_p^{(1)}(s)\mathrm{e}^{-s}s\;\mathrm{d}s = \delta_{q,p} \tag{16}$$

where $s = 2\alpha\omega$.

Here we observe an interesting property: since a Laguerre polynomial $L_q^{(1)}$ is a polynomial of order q, each spectrum in this orthogonal set is given by a polynomial of order $(q + 1)$. Consequently, the orthogonal set does not contain the zeroth-order wave given by $f_n(\omega) = \mathrm{e}^{-\alpha\omega}$, i.e., the fundamental X wave. This is due to the divergent factor $1/k_\perp \propto 1/\omega$ in the Fourier representation of the wave [13]. Although it actually cancels out in the $\int_0^\infty k_\perp\,\mathrm{d}k_\perp$ integration, this factor causes the spectrum of the wave to diverge for low frequencies. It is possible to find a set of waves with higher order spectra that are all orthogonal to the 'fundamental' wave but they do not contain the polynomial of degree 1. Suppose that such a polynomial exists. Then the orthogonality relation with the fundamental wave reads

$$\int_0^\infty p_1(\omega)\mathrm{e}^{-2\alpha\omega}\omega^{-1}\,\mathrm{d}\omega. \tag{17}$$

This diverges at the origin, except for $p_1(\omega) = C\omega$. In the latter case, we have

$$C\int_0^\infty \mathrm{e}^{-2\alpha\omega}\,\mathrm{d}\omega = \frac{C}{2\alpha} \neq 0. \tag{18}$$

It follows, therefore, that there are no such first-order polynomial X waves that would be orthogonal to the fundamental wave. (Note the choice of the spectrum in [14].)

Using spectra of the form given in equation (15), the orthogonal set of X waves assumes the form

$$\Phi_{n,q}^{\mathrm{ort}}(\boldsymbol{r},t) = \sum_{l=0}^{q}\frac{(-1)^l}{\sqrt{q+1}}\binom{q+1}{q-l}\frac{(2\alpha)^{l+1}}{l!}\Phi_{n,l+1}(\boldsymbol{r},t). \tag{19}$$

Consequently, they form an orthogonal basis for the (linear space of) X waves from which the waves of spectral order zero have been excluded. For illustrations of the first four orthogonal wave forms, see figure 2.

5. Applications of orthogonal X waves

In this section we discuss some potential applications of orthogonal X waves.

5.1. Bessel-X waves

Pulsed optical Bessel beams, or Bessel-X waves [8], are ultrashort (∼fs) nondiffracting light pulses whose spectrum is approximately represented as [5]

$$S(\omega) = S_0\sqrt{\frac{\omega}{\omega_0}}\mathrm{e}^{-\tau^2(\omega-\omega_0)^2} \tag{20}$$

where ω_0 is the carrier frequency, τ^{-1} characterizes the spectral width of the pulse and S_0 is a constant. Here we use orthogonal X waves to approximate the Bessel-X wave (denoted as Φ_{BX}):

$$\Phi_{\mathrm{BX}}(\boldsymbol{r},t) = \sum_{q=0}^{N} c_q\Phi_{0,q}^{\mathrm{ort}}(\boldsymbol{r},t). \tag{21}$$

We note that the above expansion coefficients c_q depend on the attenuation constant α which is contained in $\Phi_{n,q}^{\mathrm{ort}}(\boldsymbol{r},t)$. In figure 3, we show a Bessel-X wave spectrum for the carrier

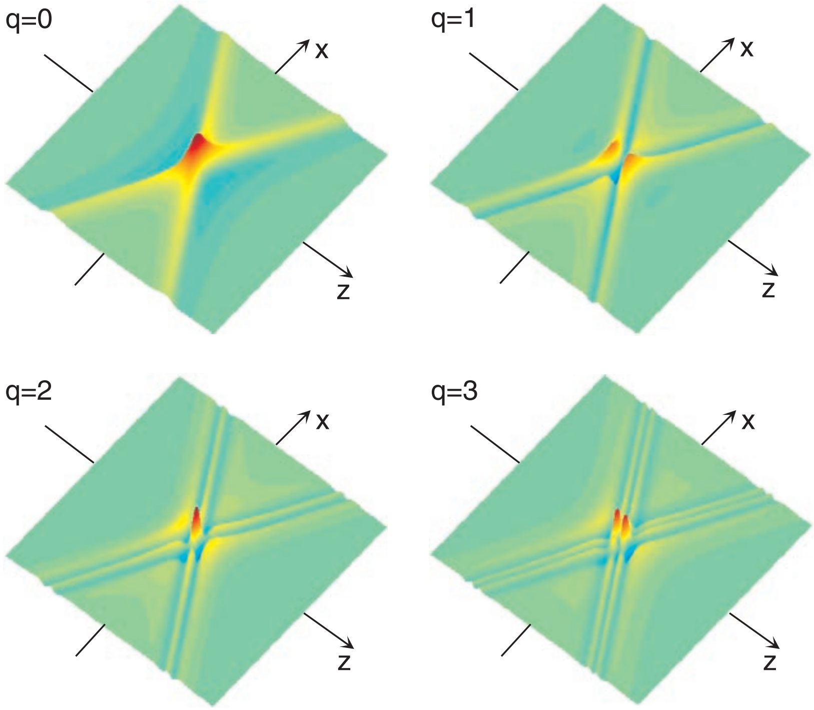

Figure 2. First four orthogonal X waves, $\Phi^{\mathrm{ort}}_{n,q}$. Orthogonal X waves feature an increasing number of 'halo wave fronts' (or 'halo toroids' [8]), required for making the scalar product between different wave modes vanish.

(angular) frequency $\omega_0 = 3.14 \times 10^{15}\ \mathrm{s}^{-1}$ and pulse duration (FWHM) 3 fs, together with its approximants having 5, 10 and 15 terms.

In figure 3, the approximation with the first five terms shows no clear resemblance to the actual spectrum, while that with 15 terms appears physically adequate. Very good agreement is achieved already for 20 terms. The Bessel-X pulse containing 20 orthogonal components is shown in figure 4. The number of terms strongly depends on the spectral width and the carrier frequency of the Bessel-X spectrum and it is smallest when the two are of the same magnitude. Hence, the orthogonal expansion using $\Phi^{\mathrm{ort}}_{n,q}$ is useful for ultrashort pulses which can be used to derive analytic (though approximate) expressions.

5.2. Detection and generation

From the practical point of view, the question arises how to produce and measure orthogonal X waves. First, we point out that the orthogonality condition was derived based on the signal in all space at a fixed time. Physical measurements, on the other hand, are usually performed using a planar detector, with a time-dependent measurement result. This, however, does not affect the orthogonality since nondiffracting waves only depend on $(z - vt)$; therefore, the spatial z integration can always be replaced with a temporal integration at fixed z.

Another physical observation is that to experimentally detect different orthogonal components carried by an X wave, amplitude information is needed instead of intensity. Hence, in the range of optical frequencies, interferometric measurement is needed. For radio frequencies where nondiffracting waves have recently been produced, both with axicons [15]

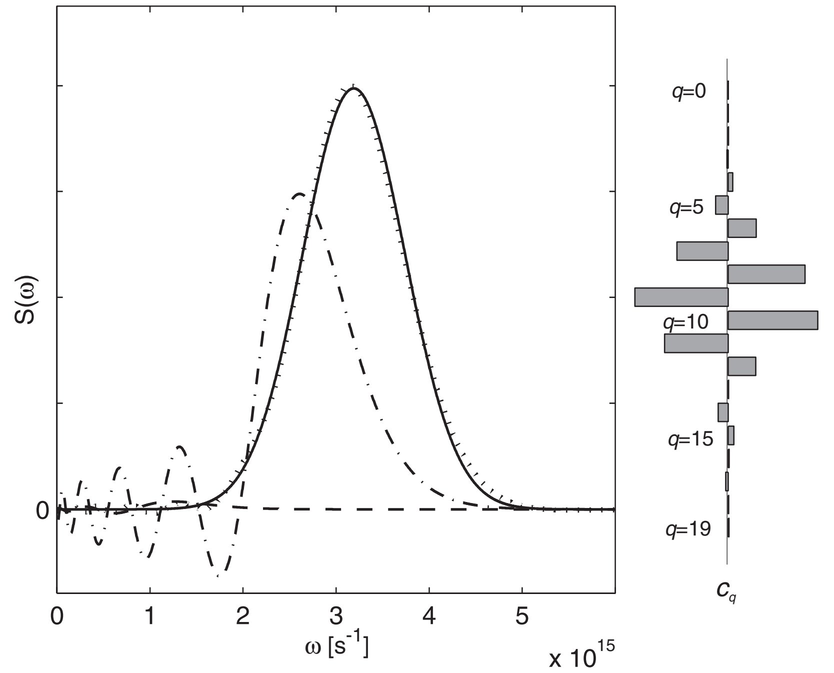

Figure 3. Approximate representations for a Bessel-X pulse spectrum: (left) solid line shows the exact Bessel-X wave spectrum while orthogonal representations of ascending order are shown by the dashed line ($N = 5$), the dashed–dotted line ($N = 10$) and the dotted line ($N = 15$). An orthogonal expansion for $N \geqslant 20$ does not visibly differ from $S(\omega)$. (Right) relative weights c_q of different orthogonal components. Here, $\alpha = 6.2$ fs; this value was chosen for optimal convergence.

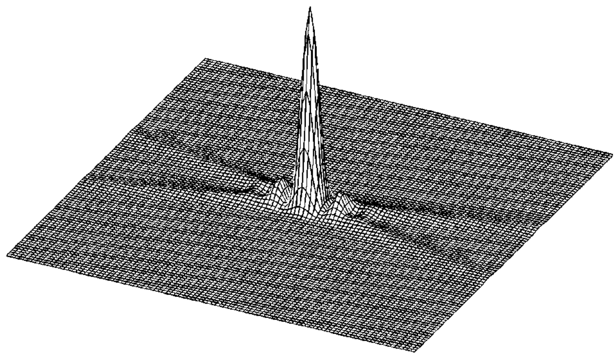

Figure 4. Intensity of the Bessel-X pulse as given by equation (21) with 20 terms to be compared with figure 1(a) in [5]. The pulse duration (FWHM) is 3 fs, carrier frequency $\omega_0 = 3.14 \times 10^{15}\ \mathrm{s}^{-1}$ and $\alpha = 6.2$ fs. The area illustrated in this meridional plane is $20 \times 20\,\mu\mathrm{m}^2$.

and computer holograms [16], phase-sensitive measurements are easily carried out and an experimental distinction between the different wave modes is feasible.

Finally, the generation of orthogonal wave modes requires sensitive spatiotemporal control of the emitted field and, as such, poses challenges for the optical instrumentation. However, novel methods have been developed, for example, to generate focus-wave modes [17], and similar techniques can also be used for the new X waves.

5.3. X-wave transform

Finally, we would like to compare the orthogonality of X waves and the newly presented X-wave transform [10]. The X-wave transform provides a general way of representing ordinary (diffracting) waves using a superposition of nondiffracting waves of different cone angles ζ, azimuthal orders n and spectra. As we have also concluded, the X-wave components with differents ζ and n are orthogonal by construction, which is already intrinsically used in an X-wave transform. However, heretofore, no orthogonal expansion has been given in the literature for the spectrum within the X-wave transform and, if such is needed, orthogonal X waves provide a tool for that purpose.

6. Conclusions

We have presented an orthogonality condition for nondiffracting waves and derived an orthogonal set of (polynomial) X waves, which can be used to expand other nondiffracting waves in terms of analytically known solutions. In particular, we have demonstrated that a Bessel-X wave-type spectrum can be expressed with orthogonal X wave spectra and, consequently, the Bessel-X wave itself can be represented as a sum of orthogonal X waves.

Recently, applications have been proposed for pulsed optical X waves both in the context of image transmission [5] and optical communications [9]. We expect that both of these fields could benefit from the possibility of using spatially orthogonal pulse forms which allow separate transmission channels.

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Appendix. Derivation of the spectral scalar product for X waves

The three-dimensional (spatial) scalar product of two nondiffracting wave fields at the same instant of time, t, is given by equation (5). By substituting the Fourier representations of the waves (equation (1)) it assumes the form

$$\begin{aligned}\langle \Phi_F \mid \Phi_G \rangle = \frac{1}{2\pi} \int_0^\infty \mathrm{d}\omega_1 \int_{-\infty}^\infty \mathrm{d}k_{z1} \int_0^\infty k_{\perp 1} \mathrm{d}k_{\perp 1} \int_0^{2\pi} \mathrm{d}\beta_1\, \tilde{\Phi}_F^*(\boldsymbol{k}, \omega) \mathrm{e}^{\mathrm{i}\omega_1 t} \\ \times \int_0^\infty \mathrm{d}\omega_2 \int_{-\infty}^\infty \mathrm{d}k_{z2} \int_0^\infty k_{\perp 2} \mathrm{d}k_{\perp 2} \int_0^{2\pi} \mathrm{d}\beta_2\, \tilde{\Phi}_G(\boldsymbol{k}, \omega) \mathrm{e}^{-\mathrm{i}\omega_2 t} \\ \times \delta(k_{\perp 1} \cos\beta_1 - k_{\perp 2} \cos\beta_2) \delta(k_{\perp 1} \sin\beta_1 - k_{\perp 2} \sin\beta_2) \delta(k_{z1} - k_{z2}). \end{aligned} \tag{A.1}$$

Integrating all the delta functions (also those within the $\tilde{\Phi}$), the scalar product becomes

$$\langle \Phi_F | \Phi_G \rangle = \frac{c}{2\pi \cos\zeta_G} \left| 1 - \frac{\tan\zeta_G}{\tan\zeta_F} \right|^{-1} \int_0^{2\pi} \mathrm{d}\beta_1 \left[\omega_1 F^*(\omega_1, \beta_1) G(\omega_1, \beta_1) \right]_{\omega_1 = 0} \tag{A.2}$$

if the waves have different velocities of propagation, i.e., different cone angles. The fact that only the zero-frequency component contributes to the scalar product may be understood as follows. In the wave vector space, the Fourier representation of the nondiffracting wave field is strictly confined to a cone that opens with an angle given by ζ around the k_z-axis. If two waves have different cone angles, the overlap of their Fourier support is simply the origin, and it is met for $\omega = 0$. Within optics, this corresponds to the dc component of the field and has, therefore, no physical consequences. Furthermore, for all orthogonal X waves discussed in this paper, the spectrum $F(\omega, \beta)$ is bounded for $\omega \to 0$ and the scalar product vanishes. Another interpretation for this result is that in real space, most of the wave energy lies on the cones of propagation which do not overlap for waves with different velocities. If, however, the wave spectrum $F(\omega, \beta)$ diverges for $\omega = 0$ (or, equivalently, the Fourier spectrum $f_n(\omega)$ does not tend to zero for $\omega \to 0$), the wave energy is not bound to the cone of propagation and the integral in the scalar product is dominated by the asymptotic domain away from the axis of propagation.

On the other hand, if the two waves propagate with the same velocity, their scalar product is

$$\begin{aligned}\langle \Phi_F | \Phi_G \rangle &= \frac{c}{2\pi \cos\zeta} \int_0^\infty \mathrm{d}\omega_1 \int_0^\infty k_{\perp 1}\,\mathrm{d}k_{\perp 1} \\ &\quad\times \int_0^{2\pi} \mathrm{d}\beta_1\, F^*(\omega_1, \beta_1) G(\omega_1, \beta_1) \delta^2(k_{\perp 1} - \omega_1 \sin\zeta / c) \\ &= \frac{R\tan\zeta}{2\pi^2} \int_0^\infty \omega_1\,\mathrm{d}\omega_1 \int_0^{2\pi} \mathrm{d}\beta_1 F^*(\omega_1, \beta_1) G(\omega_1, \beta_1). \end{aligned} \tag{A.3}$$

Here the integration is formally extended only over a cylinder of radius R since nondiffracting waves are not square integrable over the entire space. If the integral involved yields zero, the waves are orthogonal for $R \to \infty$, whereas, in the opposite case, this limit results in a divergence reflecting the infinite energy (and divergent L_2 norm) of ideal nondiffracting waves.

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